

## Problem Set 10: Due Friday, October 7

**Problem 1:** Consider the unique linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with

$$[T] = \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix}.$$

Find, with explanation, vectors  $\vec{u}, \vec{w} \in \mathbb{R}^2$  with  $\text{Null}(T) = \text{Span}(\vec{u})$  and  $\text{range}(T) = \text{Span}(\vec{w})$ .

**Problem 2:** Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation. Recall that

$$\text{Null}(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}.$$

- Show that if  $\vec{v}_1, \vec{v}_2 \in \text{Null}(T)$ , then  $\vec{v}_1 + \vec{v}_2 \in \text{Null}(T)$ .
- Show that if  $\vec{v} \in \text{Null}(T)$  and  $c \in \mathbb{R}$ , then  $c \cdot \vec{v} \in \text{Null}(T)$ .

**Problem 3:** Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the unique linear transformation with

$$[T] = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}.$$

Explain why  $T$  has an inverse and calculate

$$T^{-1} \left( \begin{pmatrix} 5 \\ 1 \end{pmatrix} \right).$$

**Problem 4:** Consider the following system of equations:

$$\begin{array}{rclcl} x & + & 4y & = & -3 \\ 2x & + & 5y & = & 8. \end{array}$$

- Rewrite the above system in the form  $A\vec{v} = \vec{b}$  for some matrix  $A$  and vector  $\vec{b}$ .
- Explain why  $A$  is invertible and calculate  $A^{-1}$ .
- Use  $A^{-1}$  to solve the system.

**Problem 5:** In this problem, let  $0$  denote the  $2 \times 2$  zero matrix, i.e. the  $2 \times 2$  matrix where all four entries are  $0$ .

- Give an example of a nonzero  $2 \times 2$  matrix  $A$  with  $A \cdot A = 0$ .
- Show that if  $A$  is invertible and  $A \cdot A = 0$ , then  $A = 0$ .

*Note:* Since  $0$  is not invertible, it follows from part b that there is no invertible matrix  $A$  with  $A \cdot A = 0$ .

**Problem 6:** Let  $A, B, C$  all be invertible  $2 \times 2$  matrices. Must there exist a  $2 \times 2$  matrix  $X$  with

$$A(X + B)C = I?$$

Either justify carefully or give a counterexample.