Homework 3: Due Friday, February 12

Note: Throughout this assignment, let f_n be the the sequence of Fibonacci numbers, i.e. define $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$.

Exercises

Exercise 1: A friend tries to convince you that $2 \mid f_n$ for all $n \geq 3$. Here is their argument using strong induction. For the base case, notice that $f_3 = 2$, so $2 \mid f_3$. For the inductive step, suppose that the $n \geq 4$ and we know the result for all k with $3 \leq k < n$. Since $f_n = f_{n-1} + f_{n-2}$ and we know by induction that $2 \mid f_{n-1}$ and $2 \mid f_{n-2}$, it follows that $2 \mid f_n$. Therefore, $2 \mid f_n$ for all $n \geq 3$.

Now you know that your friend's argument must be wrong because $f_7 = 13$ and $2 \nmid 13$. Pinpoint the fundamental error. Be as explicit and descriptive as you can.

Exercise 2: Find a simpler way to describe the set $\{101k + 82\ell : k, \ell \in \mathbb{Z}\}$.

Interlude: We saw how fast the Euclidean Algorithm ran in class. However, it is not obvious why the algorithm terminates so quickly. In the next exercise, we work to understand the theory behind the speed. Let $a, b \in \mathbb{N}$ with $b \neq 0$. Write $a = qb + r$ where $q, r \in \mathbb{N}$ and $0 \leq r < b$. Notice that after one step of the algorithm, the new second argument r may not be much smaller than the original second argument b. For example, if $a = 77$ and $b = 26$, then we have $q = 2$ and $r = 25$. However, it turns out that after *two* steps of the Euclidean Algorithm, the new second argument will be at most half the size of the original second argument. This is what we will prove in the next exercise. From this fact, it follows that on input $(a, b) \in \mathbb{N}^2$, the algorithm terminates in at most $2\log_2 b$ many steps.

Exercise 3: Let $a, b \in \mathbb{N}$ with $b \neq 0$. In the first step of the algorithm, we write $a = qb + r$ where $q, r \in \mathbb{N}$ and $0 < r < b$ (we can assume that $r \neq 0$ because otherwise the algorithm stops at the next step). In the next step of the algorithm, we write $b = pr + s$ where $p, s \in \mathbb{N}$ and $0 \leq s < r$. Show that $s < \frac{b}{2}$.
Hint: You may find it useful to break the problem into cases based on how large r is.

Problems

Problem 1: Let $r \in \mathbb{R}$ with $r \neq 1$. Use induction to show that

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all $n \in \mathbb{N}$.

Problem 2:
a. Show that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

for all $n \in \mathbb{N}^+$.

b. Use part (a) and some linear algebra to prove that

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$

for all $n \in \mathbb{N}^+$.

Problem 3: Show that for all $a \in \mathbb{Z}$, either there exists $k \in \mathbb{Z}$ with $a^2 = 3k$ or there exists $k \in \mathbb{Z}$ with $a^2 = 3k + 1$. In other words, show that for all $a \in \mathbb{Z}$, the unique remainder upon dividing a^2 by 3 is always either 0 or 1.
Hint: Start by performing division with remainder on a.

Problem 4: Use the Euclidean Algorithm to find the greatest common divisor of the following pairs of numbers a and b. Furthermore, once you find the greatest common divisor m, find $k, \ell \in \mathbb{Z}$ such that $ka + \ell b = m$.

- $a = 234$ and $b = 165$.
- $a = 562$ and $b = 471$.

Problem 5: Show that $\gcd(f_{n+1}, f_n) = 1$ for all $n \in \mathbb{N}$.

Problem 6: Show that $\{n \in \mathbb{Z} : \gcd(n, n+2) = 2\} = \{2n : n \in \mathbb{Z}\}$.