

## Homework 10 : Due Monday, September 27

**Problem 1:** Let  $G$  be a cyclic group of order  $n \in \mathbb{N}^+$  and let  $k \in \mathbb{Z}$  satisfy  $\gcd(k, n) = 1$ . Define  $f: G \rightarrow G$  by  $f(x) = x^k$ . Show that  $f$  is a permutation of  $G$ .

**Problem 2:** Let  $n \in \mathbb{N}^+$ .

- Given  $k \geq 2$ , find a formula for the number of  $k$ -cycles in  $S_n$  and explain why it is correct. (Remember that  $(1\ 2\ 3) = (2\ 3\ 1)$  so don't count it twice.)
- Find a formula for the number of permutation in  $S_n$  which are the product of two disjoint 2-cycles and explain why it is correct.
- In the special case of  $n = 5$ , calculate the number of permutations of  $S_5$  of each cycle type (so you should explicitly calculate the number of 4-cycles, the number of permutations which are the product of a 3-cycle and 2-cycle which are disjoint, etc.). Notice that all of your answers divide  $|S_5| = 120$ . This is not an accident, as we will see later.

**Problem 3:** Let  $n \in \mathbb{N}$  with  $n \geq 2$ . We say that a set  $B \subseteq S_n$  *generates*  $S_n$  if the smallest subgroup of  $S_n$  containing  $B$  is  $S_n$  itself. In other words,  $B$  generates  $S_n$  if every element of  $S_n$  can be written as a product of elements of  $B$  and their inverses. For example, if  $B$  is the set of all transpositions, then we know from class that  $B$  generates  $S_n$ .

- Show that  $\{(1\ a) : 2 \leq a \leq n\}$  generates  $S_n$ .
- Show that  $\{(a\ a+1) : 1 \leq a \leq n-1\}$  generates  $S_n$ .
- Show that  $\{(1\ 2), (1\ 2\ 3\ \cdots\ n)\}$  generates  $S_n$ .

*Hint:* Don't reinvent the wheel every time. You already know you can get everything from the transpositions. Once you've done part a, you know you can get everything from that smaller set, etc.

**Problem 4:** Recall that given a group  $G$ , we defined

$$Z(G) = \{a \in G : ga = ag \text{ for all } g \in G\}$$

You showed in Homework 9 that  $Z(G)$  is always a subgroup of  $G$ . This subgroup is called the *center* of  $G$ . Show that  $Z(S_n) = \{id\}$  for all  $n \geq 3$ .

*Hint:* You need to take an arbitrary  $\sigma \in S_n$  with  $\sigma \neq id$  and show that there exists an element of  $S_n$  which does not commute with  $\sigma$ . I recommend that you avoid cycle notation and just work with  $\sigma$  as a function.