

## Homework 6: Due Wednesday, March 15

**Problem 1:** Suppose that  $H$  and  $K$  are both normal subgroups of  $G$ . Show that  $H \cap K$  is a normal subgroup of  $G$ .

**Problem 2:** Let  $G$  be a group and let  $H$  be a normal subgroup of  $G$ .

- Show that  $G/H$  is abelian if and only if  $a^{-1}b^{-1}ab \in H$  for all  $a, b \in G$ .
- Suppose  $[G : H]$  is finite and let  $m = [G : H]$ . Show that  $a^m \in H$  for all  $a \in G$ .

**Problem 3:** Let  $G$  and  $H$  be groups. Show that  $G \times H \cong H \times G$ .

**Problem 4:** Determine, with proof, whether the following pairs of groups are isomorphic.

- $A_6$  and  $S_5$ .
- $\mathbb{Z}/84\mathbb{Z}$  and  $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$ .
- $U(\mathbb{Z}/18\mathbb{Z})$  and  $\mathbb{Z}/6\mathbb{Z}$ .
- $S_4$  and  $\mathbb{Z}/6\mathbb{Z} \times U(\mathbb{Z}/5\mathbb{Z})$ .
- $A_4$  and  $D_6$ .
- $U(\mathbb{Z}/5\mathbb{Z})$  and  $U(\mathbb{Z}/10\mathbb{Z})$ .
- $S_3 \times \mathbb{Z}/2\mathbb{Z}$  and  $A_4$ .
- $D_4/Z(D_4)$  and  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

**Problem 5:** Consider the group  $G = \mathbb{R} \setminus \{-1\}$  with operation  $a * b = a + b + ab$  from Homework 3. Let  $H$  be the group  $\mathbb{R} \setminus \{0\}$  with operation equal to the usual multiplication. Show that  $G \cong H$ .

**Problem 6:** Consider the group  $G = U(\mathbb{Z}/15\mathbb{Z})$ . Find nontrivial cyclic subgroups  $H$  and  $K$  of  $G$  such that  $G$  is the internal direct product of  $H$  and  $K$ . Use this to find  $m, n \in \mathbb{N}$  with  $m, n \geq 2$  such that  $U(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ .